Math 3235 Probability Theory

3/28/23

Bivariate normal distribution

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}}$$

$$\exp\left(-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)\right)$$

$-1 < \rho < 1$

Bivariate Standard Normal distribution

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy =$$

$$\int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)\right] dy =$$

$$\int_{-\infty}^{\infty} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy = \sqrt{2\pi}\,\sigma \qquad (1)$$

$$(y^2 - 2\rho xy + \rho x^2) = (y - \rho x)^2$$

$$(x^2 - 2\rho xy + y^2) = (y - \rho x)^2 + (1-\rho^2)x^2$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[(y-\rho x)^2 + (1-\rho^2)x^2\right]\right) dy$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(y-\rho x)^2}{2(1-\rho^2)}\right) dy}_{(1) \text{ with } \mu = \rho x,\ \sigma^2 = (1-\rho^2)}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(y-\rho x)^2}{2(1-\rho^2)}\right)$$

$$\mathbb{E}(Y|X=x) = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)\,dy =$$

$$= \rho x$$

$$\mathbb{E}(XY) = \rho$$

$$\mathbb{E}(X) = \mathbb{E}(Y) = 0$$

$$\mathrm{Var}(X) = \mathrm{Var}(Y) = 1$$

$$\rho = \mathrm{corr}(X,Y) = \rho_{X,Y}$$

If X and Y are bivariate standard Normal Then

$$X \perp\!\!\!\perp Y \iff \rho_{X,Y} = 0$$

If X and Y are bivariate Normal $\iff$ $aX + bY$ is a Normal r.v. for any a, b real.

Moments

$$m_k = \mathbb{E}(X^k)$$

k-th moment.

$$m_1 = \mathbb{E}(X)$$

$$\text{var}(X) = m_2 - m_1^2$$

1) Not all distribution have all moments!!

X is a Cauchy r.v.

$\mathbb{E}(X^k)$ exists if and only if $K < 1$ $(K = 0)$

2) X is a r.v. and you know $m_K = \mathbb{E}(X^k)$ for every $K \geq 0$. Can you reconstruct the p.d.f. of X?

If the power series

$$\sum_{K=0}^{\infty} \frac{t^k}{k!} \mathbb{E}(X^k)$$

converges for $|t| < \delta$, $\delta > 0$

Then the moment uniquely define

The p.d.f. of X.

Moment generating function.

If X is continuous

$\mathbb{E}(s^X)$ not "nicely" defined

$s = e^t$

$$M_X(t) = \mathbb{E}\left(e^{tX}\right)$$

$M_X(t)$ is called the m.g.f. of X.

Sum of indep. r.v.:

X, Y indep.

$$\mathbb{E}(e^{t(X+Y)}) = \mathbb{E}(e^{tX} e^{tY}) =$$

$$\mathbb{E}(e^{tX})\,\mathbb{E}(e^{tY})$$

$$M_{X+Y}(t) = M_X(t)\, M_Y(t)$$

Why the name?

$$M_X(0) = \mathbb{E}(e^{0 \cdot X}) = 1$$

$$\frac{d}{dt} M_X(t) = \mathbb{E}(X e^{tX})$$

$$\frac{d}{dt} M_X(0) = \mathbb{E}(X)$$

$$\frac{d^2}{dt^2} M_X(t) = \mathbb{E}(X^2 e^{tX})$$

$$\frac{d^2}{dt^2} M_X(0) = \mathbb{E}(X^2)$$

$$\vdots$$

$$\frac{d^k}{dt^k} M_X(0) = \mathbb{E}(X^k)$$

Theorem: if $M_X(t)$ of X exists and is finite for $|t| < \delta$, $\delta > 0$

Then there is a unique distribution $f(x)$ with m.g.f. $M_X(t)$.

Moreover we have

$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbb{E}(X^k)$$

Ex. 7.9.

$$f(x) = \begin{cases} \frac{1}{x\sqrt{2\pi}} \exp\left[-\frac{1}{2}(\log x)^2\right] & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_a(x) = [1 + a \sin(2\pi \log x)] f(x)$$

$-1 \leq a \leq 1$ (to be a p.d.f.)

$$\int_{-\infty}^{\infty} x^k f_a(x)dx = \int_{-\infty}^{\infty} x^k f(x)\, dx \quad \forall a$$

X is Normal STandard

$$M_X(t) = \mathbb{E}\left(e^{tX}\right) =$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} f(x)\, dx =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{x^2}{2}} dx =$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} e^{-\left(\frac{t^2}{2} - tx + \frac{x^2}{2}\right)} dx =$$

$$= e^{\frac{t^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2} dx =$$

$$= e^{\frac{t^2}{2}} \cdot M_X(t)$$

$$M_X'(t) = t e^{\frac{t^2}{2}}$$

$$M_X''(t) = e^{\frac{t^2}{2}} + t^2 e^{\frac{t^2}{2}}$$

$$M_X^{(k)}(t) = P_k(t) e^{\frac{t^2}{2}}$$

$P_k(t)$ is a polynomial of degree k.

$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbb{E}(X^k)$$

Taylor series of $e^{\frac{t^2}{2}}$

$$e^{\frac{t^2}{2}} = \sum_{k=0}^{\infty} \frac{t^{2k}}{2^k} \frac{1}{k!}$$

$$e^{\frac{t^2}{2}} = \sum_{k=0}^{\infty} a_k t^k$$

k odd $\quad a_k = 0$

$k = 2n$ even $\quad a_k = \frac{1}{2^n} \frac{1}{n!}$

Comparing with the expression for $M_X(t)$ we get

$$\mathbb{E}(X^{2n+1}) = 0$$

$$\mathbb{E}(X^{2n}) = \frac{(2n)!}{2^n n!}$$